Math 247A Lecture 25 Notes

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1 Oscillatory Integrals in Higher Dimensions

1.1 Nonstationary phase

Here is the case of nonstationary phase.

Proposition 1.1. *Let* $\phi : \mathbb{R}^d \to \mathbb{R}$, $\psi : \mathbb{R}^d \to \mathbb{C}$ *be smooth. Assume* $\operatorname{supp} \psi$ *is compact and* $\|\nabla \phi(x)| \neq 0$ *for all* $x \in \operatorname{supp} \psi$. *Then* $I(\lambda) = \int e^{i\lambda\phi(x)} \psi(x)\, dx$ *satisfies*

$$|I(\lambda)| \lesssim_m \lambda^{-m} \qquad \forall m \geq 0.$$

Proof. As in the 1 dimensional case, we use integration by parts. We write

$$e^{i\lambda\phi(x)} = \frac{\nabla \phi(x)}{i\lambda |\nabla \phi(x)|^2} \cdot \nabla(e^{i\lambda\phi(x)}).$$

Then

$$I(\lambda) = \int e^{i\lambda\phi(x)} \nabla \cdot \left[\frac{\nabla \phi(x)}{i\lambda |\nabla \phi(x)|^2} \psi(x) \right] dx,$$

so

$$|I(\lambda)| \lesssim \lambda^{-1},$$

where the implicit constant depends on the C^2 norm of ϕ and the C^1 norm of ψ. Now iterate. □

There is an equivalent of Van der Corput's lemma.

Proposition 1.2. *Let* $\phi : \mathbb{R}^d \to \mathbb{R}$, $\psi : \mathbb{R}^d \to \mathbb{C}$ *be smooth. Assume* ψ *is compactly supported and* $|D^\alpha \phi(x)| \geq 1$ *for al* $x \in \operatorname{supp} \psi$ *for some* $\alpha \in \mathbb{N}^d$ *with* $|\alpha| \geq 1$. *Then* $I(\lambda) = \int e^{i\lambda\phi(x)} \psi(x)\, dx$ *satisfies*

$$|I(\lambda)| \leq C(|\alpha|, \phi) \lambda^{-1/|\alpha|} [\|\psi\|_\infty + \|\nabla \psi\|_1].$$

Remark 1.1. This is worse than the previous proposition when $|\alpha| = 1$. We will also beat it when $|\alpha| = 2$, so we will not actually prove it.

1.2 Stationary phase and Moore's change of variables lemma

Here is the case of stationary phase.

Proposition 1.3 (stationary phase). *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be smooth, and assume ϕ has a nondegenerate critical point at x_0; that is, $\nabla\phi(x_0) = 0$, but* $\det \left[\frac{\partial^2 \phi}{\partial x_i x_j}\right]_{1 \le i,j \le d}(x_0) \neq 0$. *Assume that $\psi : \mathbb{R}^d \to \mathbb{C}$ is smooth and supported in a sufficiently small neighborhood of x_0. Then*

$$\begin{aligned} I(\lambda) &= \int e^{i\lambda\phi(x)}\psi(x)\,dx \\ &= e^{i\lambda\phi(x_0)}\psi(x_0)(2\pi i)^{d/2}\lambda^{-d/2}(\det[D^2\phi(x_0)])^{-1/2} + O(\lambda^{-d/2-1}) \end{aligned}$$

as $\lambda \to \infty$.

Remark 1.2. If we just aim for the correct decay order (and not the precise coefficient), we argue as follows: Let $a : \mathbb{R}^d \to \mathbb{R}$ be a cutoff with

$$a(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| \ge 2 \end{cases}$$

and decompose $I(\lambda) = I_1(\lambda) + I_2(\lambda)$, where

$$I_1(\lambda) = \int e^{i\lambda\phi(x)}\psi(x)a(\lambda^{1/2}(x - x_0))\,dx.$$

Then

$$|I_1(\lambda)| \lesssim \lambda^{-d/2}.$$

Integration by parts gives

$$|I_2(\lambda)| \lesssim_m \lambda^{-m} \qquad \forall m \ge 0.$$

Lemma 1.1 (Morse). *If x_0 is a nondegenerate critical point of a smooth function $\phi : \mathbb{R}^d \to \mathbb{R}$, then there exists a smooth change of variables $x \mapsto y(x)$ such that $y(x_0) = 0$, $\frac{\partial y}{\partial x}(x_0) = \mathrm{Id}$, and*

$$\phi(x) - \phi(x_0) = \sum_{j=1}^{d} \frac{1}{2}\lambda_j y_j^2,$$

where $\lambda_1, \dots, \lambda_d$ are the eigenvalues of $D^2\phi(x_0)$.

Proof. Performing an orthogonal change of variables, we may assume that $D^2\phi(x_0) = \mathrm{diag}(\lambda_1, \dots, \lambda_d)$. By Taylor expansion,

$$\phi(x) = \phi(x_0) + \cancelto{0}{\nabla\phi(x_0)} \cdot (x - x_0) + \int_0^1 (1 - t)\frac{d^2}{dt^2}[\phi(x_0 + t(x - x_0))]\,dt.$$

So

$$\begin{aligned}\phi(x)-\phi(x_0) &= \int_0^1 (1-t)\frac{d}{dt}[(x-x_0)\cdot\nabla\phi(x_0+t(x-x_0))]\,dt\\ &= \sum_{i,j\geq 1}\int_0^1 (1-t)(x-x_0)_i(x-x_0)_j\frac{\partial^2\phi}{\partial x_i\partial x_j}(x_0+t(x-x_0))\,dt\\ &= \sum_{i,j\geq 1}(x-x_0)_i(x-x_0)_j m_{i,j}(x),\end{aligned}$$

where $m_{i,j}(x)=\int_0^1(1-t)\frac{\partial^2\phi}{\partial x_i\partial x_j}(x_0+t(x-x_0))\,dt$. Note that the $m_{i,j}$ are smooth, $m_{i,j}(x)=m_{j,i}(x)$, and $m_{i,j}(x_0)=\frac{1}{2}\frac{\partial^2\phi}{\partial x_i\partial x_j}(x_0)$. So

$$[m_{i,j}(x_0)]_{1\leq i,j\leq d}=\frac{1}{2}\operatorname{diag}(\lambda_1,\ldots,\lambda_d).$$

We argue inductively. Assume

$$\phi(x)-\phi(x_0)=\frac{1}{2}\lambda_1 y_1^2+\cdots+\frac{1}{2}\lambda_{r-1}y_{r-1}^2+\sum_{i,j\geq r}\widetilde{m}_{i,j}(y)y_iy_j$$

for some $1\leq r\leq d$, where $y(x_0)$ =0, $\frac{\partial y}{\partial x}(x_0)=\mathrm{Id}$, and $\widetilde{m}_{i,j}=\widetilde{m}_{j,i}$. We know that $D^2[\mathrm{RHS}(x)]_{x=x_0}=\operatorname{diag}(\lambda_1,\ldots,\lambda_d)$. Then

$$\begin{aligned}\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{1}{2}\lambda_k y_k^2\right)\bigg|_{x=x_0} &= \left[\lambda_k\frac{\partial y_k}{\partial x_i}\frac{\partial y_k}{\partial x_j}+\lambda_k y_k\frac{\partial^2 y_k}{\partial x_i\partial x_j}\right]_{x=x_0}\\ &= \lambda_k\delta_{i,k}\delta_{j,k}.\end{aligned}$$

So

$$\left[D^2\left(\sum_{i,j\geq r}\widetilde{m}_{i,j}(y)y_iy_j\right)\right](x_0)=\operatorname{diag}(0,\ldots,0,\lambda_r,\ldots,\lambda_d).$$

We now have

$$\frac{\partial^2}{\partial x_k\partial x_\ell}\left(\sum_{i,j\geq r}\widetilde{m}_{i,j}(y)y_i,y_j\right)\bigg|_{x=x_0}=\sum_{i,j\geq r}\widetilde{m}_{i,j}(0)\left(\delta_{k,i}\delta_{\ell,j}+\delta_{\ell,i}\delta_{k,j}\right).$$

This tells us that

$$[\widetilde{m}_{i,j}(0)]_{r\leq i,j\leq d}=\frac{1}{2}\operatorname{diag}(\lambda_r,\ldots,\lambda_d)$$

Change variables as follows:

$$\begin{cases} y'_j=y_j & j\neq r\\ y'_r=\sqrt{\frac{\widetilde{m}_{r,r}(y)}{\lambda_r/2}}\left(y_r+\sum_{j\geq r+1}\frac{\widetilde{m}_{j,r}(y)}{\widetilde{m}_{r,r}(y)}y_j\right). \end{cases}$$

We need to show that this is a diffeomorphism with $y'(x_0) = 0$, $\frac{\partial y'}{\partial x}|_{x=x_0} = \mathrm{Id}$, and

$$\phi(x) = \phi(x_0) = \frac{1}{2}\lambda_1 y_1^2 + \cdots + \frac{1}{2}\lambda_r (y_r')^2 + \sum_{i,j \geq r+1} \widetilde{\widetilde{m}_{i,j}}(y) y_i y_j.$$

We have $y'(x_0) = 0$ because each y_i is 0 at x_0. For $j \neq r$,

$$\left.\frac{\partial y_j'}{\partial x_i}\right|_{x-x_0} = \delta_{i,j},$$

so

$$\left.\frac{\partial y_r'}{\partial x_i}\right|_{x=x_0} = \sqrt{\frac{\widetilde{m}_{r,r}(0)}{\lambda_r/2}} \left(\delta_{i,r} + \sum_{j \geq r+1} \cancelto{0}{\frac{\widetilde{m}_{j,r}(0)}{\widetilde{m}_{r,r}(0)}} \delta_{j,i} \right) = \delta_{i,r}$$

Now we have

$$\begin{aligned}\sum_{i,j \geq r} \widetilde{m}_{i,j}(y) y_i y_j - \frac{1}{2}\lambda_r (y_r')^2 &= \sum_{i,j \geq r} \widetilde{m}_{i,j}(y) y_i y_j \\ &\quad - \widetilde{m}_{r,r} \left(y_r^2 + 2 \sum_{j \geq r+1} \frac{\widetilde{m}_{j,r}(y)}{\widetilde{m}_{r,r}(y)} y_i, y_r \right. \\ &\quad \left. + \sum_{i,j \geq r+1} \frac{\widetilde{m}_{i,r}(y) \widetilde{m}_{j,r}(y)}{\widetilde{m}_{r,r}(y) \widetilde{m}_{r,r}(y)} y_i, y_j \right) \\ &= \sum_{i,j \geq r+1} \underbrace{\left[\widetilde{m}_{i,j}(y) - \frac{\widetilde{m}_{i,r}(y) \widetilde{m}_{j,r}(y)}{\widetilde{m}_{r,r}(y)} \right]}_{= \widetilde{\widetilde{m}}_{i,j}(y)} y_i, y_j.\end{aligned}$$

This completes the proof. $\square$